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The stability of the equilibrium of a pendulum of variable length☆

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ABSTRACT

The frequencies and modes of parametric oscillations of a pendulum of variable length for values of the modulation index from the smallest to the limit admissible values are investigated. The limits of the resonance zones of the first four oscillation modes are constructed and investigated by analytical and numerical methods, and the fundamental qualitative properties of the higher modes are established. Complete degeneracy of the modes with even numbers, i.e., coincidence of the frequencies of the odd and even eigenmodes for admissible values of the modulation parameter, is proved. The global pattern of the limits of the regions of stability of the lower position of equilibrium is constructed and it is shown that it differs considerably from the Ince–Strutt diagrams. Specific properties of the eigenmodes are established.

Despite the large number of publications devoted to problems of the instability of equilibrium states (see Refs 1-13 and the bibliography), investigations of the parametric excitation of the oscillations of multidimensional mechanical systems, including classical ones, are often limited to a theoretical or numerical analysis of the limiting cases of asymptotically small variations of the system parameters. An exception is the investigation of the oscillations of systems described by Mathieu and Meissner equations.[14,15] Small-parameter methods (Lyapunov–Poincaré[2–7]), asymptotic averaging methods (Krylov–Bogolyubov–Mitropolskii[8]), etc. are used as the mathematical techniques. Numerical investigations based on the Floquet–Lyapunov theory,[3–6] methods of mathematical physics[11–16] and the Bubnov–Galerkin–Ritz variational approaches[1,2,14,15] are also based on perturbation methods.

For applications, however, the solutions of problems of parametric stability and instability of equilibrium positions for specified values of the modulation index as small as desired are extremely important. These solutions require the development of effective numerical-analytical procedures, which can be realized using modern software for all admissible values of the parameters. The rapidly converging method of accelerated convergence in combination with the procedure of continuation with respect to the parameters enables standard and generalized parametric oscillations to be investigated over a wide and maximum admissible range of the parameters of different systems.[17–19]

Below we use an approach developed previously[17,18] to investigate periodic boundary value problems that arise when analysing the stability of the lower position of equilibrium of a plane mathematical pendulum of variable length.

1. Formulation of the problem

We will assume, for simplicity, that a holonomic mechanical system is constructed using an inextensible thread (a weightless rod) and a point mass, which is displaced along the axis in a specified (periodic) manner (release from a constraint is not admitted). It is assumed that the length of this pendulum is changed over wide acceptable limits. It is required to construct periodic modes of small oscillations, i.e., to determine the limits of the regions of instability (resonance) and stability (in the linear approximation). Using the solution of self-adjoint periodic boundary value problems in eigenvalues and eigenfunctions[16–18] it is required to construct highly accurate diagrams showing how the eigenvalues of the system depend on the modulation index within the admissible limits of its variation, by analogy to the Ince–Strutt and Meissner diagrams.[14,15]

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We will construct a mathematical model of a pendulum of variable length ℓ with a fixed axis of oscillations or rotations. The Cartesian coordinates of the point mass m ($m = 1$) are described by the expressions

$$x = \ell\sin\varphi, \quad y = -\ell\cos\varphi \quad (\mathrm{mod}\,2\pi)$$
$$\ell = \ell(t) \equiv \ell_0(1 - e\cos 2\Omega t), \quad 0 \le e < 1 \tag{1.1}$$

where φ is the angle of deflection of the pendulum axis with respect to the vertical. For simplicity we will assume that the change in the length ℓ varies harmonically with time, but the modulation index e is not assumed to be small. When the mass is displaced, the distance ℓ to the axis varies in the range $\ell_0(1 \mp e)$, i.e., it can take fairly small values for a fixed value of ℓ_0 and $e \to 1$, and also as large as desired when $\ell_0 \to \infty$ and a fixed modulation index $e < 1$. The parameters ℓ_0 – the mean distance and $2\,\Omega$ - the frequency are later eliminated by the standard procedure of changing to a dimensionless problem.

Using expressions (1.1) we will represent the potential energy U and write the coordinates of the velocity of motion $\dot{x}, \dot{y}$ of the mass, on the basis of which we will calculate the kinetic energy T. We have

$$U(\varphi,t) = -g\ell(t)\cos\varphi$$
$$\dot{x} = \ell\dot{\varphi}\cos\varphi + \dot{\ell}\sin\varphi, \quad \dot{y} = \ell\dot{\varphi}\sin\varphi - \dot{\ell}\cos\varphi$$
$$T = (\ell^2(t)\dot{\varphi}^2 + \dot{\ell}^2(t))/2, \quad \dot{\ell} = 2e\ell_0\Omega\sin 2\Omega t \tag{1.2}$$

Here g is the acceleration due to gravity. Standard calculations enable us to obtain the Lagrange equation of motion for the generalized coordinate φ

$$\ell^2\ddot{\varphi} + 2\ell\dot{\ell}\dot{\varphi} + g\ell\sin\varphi = 0, \quad \ell = \ell(t) \tag{1.3}$$

Using this equation, and also Newton's equations, we calculate the relative value of the reaction force

$$N = g\cos\varphi - \ddot{\ell} + \ell\dot{\varphi}^2, \quad N/m \to N, \quad \ddot{\ell} = 4e\ell_0\Omega^2\cos 2\Omega t$$

If the pendulum is suspended on a flexible thread, then, to retain the sign of the reaction force $N > 0$ and to avoid release from a constraint it is necessary to introduce important limitations on the quantities $\dot{\ell}$, φ, etc. In particular, when $\varphi = \dot{\varphi} \equiv 0$ the tension N of the thread is positive if $4e\ell_0\Omega^2 < g$, i.e., for a fairly small inertia force of the translational acceleration. The centrifugal forces of inertia, according to Eq. (1.3), facilitate tension, which is natural.

We will investigate the stability of the equilibrium positions $\varphi = \dot{\varphi} \equiv 0$ of system (1.3) in the linear approximation. To do this we will discard terms that are non-linear (cubic) in φ, divide Eq. (1.3) by the quantity $\ell > 0$ and introduce a new unknown variable u by the relations

$$u = \ell(t)\varphi, \quad \dot{\varphi}\ell^2(t) = \dot{u}\ell(t) - u\dot{\ell}(t)$$

We then obtain the equation

$$\ddot{u} + \ell^{-1}(t)(g - \ddot{\ell}(t))u = 0 \tag{1.4}$$

which contains three dimensional parameters g, ℓ_0, Ω and the dimensionless modulation index e. By means of replacements, which reduce the parameter μ and the argument θ to dimensionless form, we can write Eq. (1.4) as

$$u'' + \frac{\mu - 4e\pi^2\cos 2\pi\theta}{1 - e\cos 2\pi\theta}u = 0; \quad \mu = \frac{\pi^2 g}{\ell_0\Omega^2} > 0, \quad \theta = \frac{\Omega t}{\pi}; \quad 0 \le e < 1 \tag{1.5}$$

where the primes denote derivatives with respect to θ. The equation of parametric oscillations (1.5) only contains two dimensionless parameters, namely, μ and e. The argument θ has the meaning of the phases of parametric excitation, the period of which is equal to unity.

According to the general theory[2–7,14–18], it is required to obtain the values of the parameter $\mu = \mu(e)$, for which Eq. (1.5) allows of doubly periodic solutions. The curves of $\mu(e)$ bound the resonance zones, inside which there is an exponential instability of the equilibrium position and small oscillations. Outside these zones the stability conditions are satisfied to a first approximation.

The investigation of the problem of the stability of the equilibrium position of a pendulum is reduced to the form of a doubly periodic boundary value problem in eigenvalues and functions, for example, in the form of Eq. (1.5) and boundary conditions

$$u(\theta_0 + 2) = u(\theta_0), \quad u'(\theta_0 + 2) = u'(\theta_0) \tag{1.6}$$

where θ_0 is an arbitrary quantity, in particular, $\theta_0 = -1$. Conditions (1.6) are equivalent to the following[17–19]

$$u(0) = u(1) = 0, \quad \mu = \mu^s(e), \quad u = u^s(\theta, e) \tag{1.7}$$
$$u'(0) = u'(1) = 0, \quad \mu = \mu^c(e), \quad u = u^c(\theta, e) \tag{1.8}$$

Odd solutions (eigenvalues and functions) correspond to relations (1.7), while even solutions correspond to (1.8). Hence, it is required to construct solutions of two self-adjoint boundary-value problems in eigenvalues and functions with conditions of the first and second kind at the ends of the interval, i.e., to Sturm–Liouville type problems.[14–19] Note that the above formulation is non-standard, since the modulation index e is not small and occurs in the coefficients of Eq. (1.5) in a singular form: the coefficients become unbounded for $\theta = 0, 1$ as $e \to 1$. The effective numerical-analytical solution requires, as was pointed out, the development of highly accurate rapidly converging algorithms.[17,18]

In traditional formulations, the quantity e is assumed to be small, the term $2\ell\dot{\ell}\dot{\varphi}$ in Eq. (1.3) is rejected unjustifiably, while specified properties of a quantitative and qualitative nature, inherent in the mechanical problem, are ignored (see below). Attention is usually confined to Mathieu's equation, for which fundamental results have been obtained. In classical investigations, Ince–Strutt frequency diagrams have been constructed and a detailed mathematical apparatus has been developed.[2–8,14,15]

2. Approximate analytical investigation of boundary value problems

Important differences between the solutions for the initial equation (1.5) and Mathieu's equation also appear for small values of the index e, i.e., in perturbation theory. Naturally, these get worse as e increases to values of the order of unity, which is confirmed by analytical and numerical methods.

We will construct the required eigenvalues and functions of problems (1.5), (1.7) and (1.5), (1.8) in the form of regular expansions in powers of e

$$\mu^{s,c}(e) = \mu^{(0)s,c} + e\mu^{(1)s,c} + e^2\mu^{(2)s,c} + \ldots$$
$$u^{s,c}(\theta,e) = u^{(0)s,c}(\theta) + eu^{(1)s,c}(\theta) + \ldots \tag{2.1}$$

The final conditions when $\theta = 0$ and $\theta = 1$ must be satisfied identically in e, i.e., for all functions $u^{(i)s,c}(\theta)(i = 0, 1, \ldots)$. The unknowns $\mu^{(i)s,c}$, $u^{(i)s,c}$ are found by successive elementary integration of the corresponding inhomogeneous differential equations and by using the Fredholm alternative.[2,18]

At the initial step ($i = 0$, $e = 0$) elementary problems are solved; we have odd and even generating eigenfunctions and eigenvalues

$$\mu_n^{(0)s,c} = (\pi n)^2, \quad u_n^{(0)s} = A_n^s \sin \pi n\theta$$
$$u_n^{(0)c} = A_n^c \cos \pi n\theta, \quad A_n^{s,c} = \text{const}, \quad n = 0, 1, 2, \ldots \tag{2.2}$$

Using expressions (2.2) we calculate the required quantities $\mu^{(1)s,c}$, $u^{(1)s,c}$ as solutions of inhomogeneous boundary value problems; we have an equation and the Fredholm alternative

$$u^{(1)s,c''} + \mu^{(0)s,c}u^{(1)s,c} = -\mu^{(1)s,c}u^{(0)s,c} + (4\pi^2 - \mu^{(0)s,c})qu^{(0)s,c}$$
$$\mu^{(1)s,c}\int_0^1 (u^{(0)s,c})^2 d\theta = (4\pi^2 - \mu^{(0)s,c})\int_0^1 q(u^{(0)s,c})^2 d\theta; \quad q \equiv \cos 2\pi\theta \tag{2.3}$$

Substituting the known expressions (2.2) for $\mu^{(0)s,c}$, $u^{(0)s,c}$ we obtain, by the second relation of (2.3), the required expansion coefficients for the first and subsequent modes (resonance zones, see the figure)

$$\mu_1^{(1)s,c} = \mp\frac{3}{2}\pi^2, \quad \mu_n^{(1)s,c} = 0; \quad n = 2, 3, \ldots$$
$$u_1^{(1)s} = -\frac{3}{16}A_1^s \sin 3\pi\theta, \quad u_1^{(1)c} = -\frac{3}{16}A_1^c \cos 3\pi\theta$$
$$u_2^{(1)s} = B_2^s \sin 2\pi\theta, \quad u_2^{(1)c} = B_2^c \cos 2\pi\theta; \quad B_2^{s,c} = \text{const}$$
$$u_n^{(1)s} = \frac{1}{2}A_n^s\left(\frac{\sin\pi(n-2)\theta}{n^2-(n-2)^2} - \frac{\sin\pi(n+2)\theta}{(n+2)^2-n^2}\right)$$
$$u_n^{(1)c} = \frac{1}{2}A_n^c\left(\frac{\cos\pi(n-2)\theta}{n^2-(n-2)^2} - \frac{\cos\pi(n+2)\theta}{(n+2)^2-n^2}\right); \quad n \ge 3 \tag{2.4}$$

The figure shows the "eigenfrequencies" $\nu_n^{s,c} = (\mu_n^{s,c}/\pi^2)^{1/2}$.

Note that the expressions for $u_2^{(1)s,c}$ (2.4) and $u_2^{(0)s,c}$ (2.2) are identical. The equalities $\mu_n^{(1)s,c} = 0$ indicate the higher degree of dependence of $\mu_n^{s,c}$ on the index e. Subsequent approximations are found similarly, but the calculations are extremely lengthy and difficult.

A further analysis indicates that the expressions for $u_2^{(i)s,c}$ and $u_2^{(i-1)s,c}$ are also identical when $i \ge 3$. It enables us to draw a fundamental conclusion for the mechanical system considered, namely, the following expressions (see the figure) are an exact solution of the problem for $n = 2$

$$\mu_2^{s,c} = 4\pi^2, \quad u_2^s = A_2^s \sin 2\pi\theta, \quad u_2^c = A_2^c \cos 2\pi\theta; \quad u_2^{s,c''} + 4\pi^2 u_2^{s,c} = 0 \tag{2.5}$$

which are independent of the modulation index. According to solution (2.5), the eigenvalues are identical, i.e., the system is completely degenerate. Two mutually orthogonal eigenfunctions correspond to these values, and the orthogonality property is satisfied both directly for the functions $u_2^{s,c}$ and with weight $r = (1 - eq)^{-1}$, according to the theory of self-adjoint boundary value problems.[14–18] Naturally, the orthogonality condition for the functions $u_2^{s,c}$ and $u_n^{s,c}$ when $n \neq 2$ is only satisfied with this weight $r(\theta,e)$.

A calculation of higher approximations, according to expansions (2.1), enables us to refine the expressions for $\mu_1(e)$, $u_1(\theta, e)$, taking into account corrections $O(e^i)$, suitable, however, for fairly small values of $e > 0$. It follows from relations (2.4) that the quantity μ_1^c increases and μ_1^s decreases as $e > 0$ increases; their limit behaviour as $e \to 1$ requires additional investigations using analytical and numerical methods (see below).

The solution for the second resonance zone ($n=2$) is completely constructed and is represented by relations (2.5). The dependence of the functions $u_2^{s,c}$ on the index e can be introduced by normalization with weight $r(\theta,e)$; in fact, we then obtain

$$\left\|u_2^{s,c}\right\|_r^2 = 1; \quad A_2^s(e) = e\left(1-(1-e^2)^{1/2}\right)^{-1/2}, \quad A_2^c(e) = e\left((1-e^2)^{-1/2}-1\right)^{-1/2}$$
$$A_2^s(0) = A_2^c(0) = \sqrt{2}; \quad A_2^s \to 1, \quad A_2^c \to 0, \quad e \to 1 \tag{2.6}$$

Refinement of solutions (2.4) for higher oscillation modes leads to new effects, not established in existing publications. In the second approximation in e, there are no corrections for the eigenvalues $\mu_n^{s,c}$ when $n \geq 3$, i.e., they have higher orders. An analysis of the third zone, as usual, gives corrections $O(e^3)$, and they are negative and, moreover, $\mu_3^{(3)c} < \mu_3^{(3)s}$, i.e., $\mu_3^c(e) < \mu_3^s(e)$ (see the figure). This property is confirmed by calculations for all $0<e<1$. In the limit as $e \to 1$ the asymptotic $\mu_3^{s,c} \to 4\pi^2$ holds; naturally, the functions $u_3^{s,c}(\theta, e) \to u_2^{s,c}(\theta)$ (2.5). Hence, the resonance region turns out to be extremely narrow and practically degenerates into a curve, but in the region of the value $e=1$ the differences of μ_3^s and μ_3^c may be considerable – of the order of unity (see the results of calculations in Section 3 below and the figure).

An investigation of the fourth oscillation mode leads to the equality $\mu_4^{(4)s} = \mu_4^{(4)c} > 4\pi^2$, which, on further refinement, indicates that the curves coincide; $\mu_4^s(e) \equiv \mu_4^c(e)$, i.e., the fact that there is no region of resonance values similar to the case when $n=2$ (2.5). There is a resonance curve, described by the function $\mu_4(e)$, the values of which increase in the initial section $0<e<0.3$, then reach a maximum of $\mu_4=16.66$ when $e \approx 0.3$, and then decrease monotonically in the interval $0.3<e<1$ and approach the value $\mu_4=4\pi^2$ similar to the curve $\mu_3^{s,c}(e)$ (see the figure).

Another notable property is the coincidence of the resonance curves μ_n^s and μ_n^c for even values of the mode number $n=2, 4, \ldots$. This is confirmed by a more detailed analytical investigation: by the increase in the degree of approximation and the construction of the Hill determinant.[2,5,7,14,15] This qualitative result was established and proved for the first time by Kochin when solving a similar problem on the parametric oscillations of elastic crankshafts.[2,3] This mechanical effect was also confirmed using a highly accurate numerical-analytical accelerated-convergence method and numerous calculations[17–19] (see below and the figure). The behaviour of these curves as $e \to 1$ is similar to that described above. Note that the eigenfunctions $u_{2k}^{s,c}$ are orthogonal with weight $r(\theta,e)$ to one another and to the functions $u_n^{s,c}$ when $n \neq 2k$.

The resonance regions for odd mode numbers $n=3, 5, \ldots$ turn out to be extremely narrow, and they contract without limit as k increases. The curves of $\mu_n^{s,c} = (\pi n)^2 + O(e^n)$ when $e \ll 1$ and $\mu_n^{s,c} \to 4\pi^2$ as $e \to 1$, while the eigenfunctions $u_n^{s,c} \to u_2^{s,c}$ (2.5). Naturally, all these functions are mutually orthogonal with the weight indicated above; they form bases.[16–18]

Thus, we have established that the most significant region of the parameters, leading to instability, is the first one. The remaining ones are extremely narrow or degenerate.

We will now consider the problem of the stability of the parametric oscillations in the initial variable φ according to relations (1.3) and (1.4). Note that the replacement $\varphi \to u$ (1.4) is useful since it has the form of regularisation as $e \to 1$. In fact, like (1.5), (1.7) and (1.8) we will represent the Sturm–Liouville type problems in eigenvalues and functions in terms of the initial variable φ

$$(p(\theta,e)\varphi')' + \mu r(\theta,e)\varphi = 0, \quad p = r^2, \quad r = 1 - eq > 0$$
$$\varphi(0) = \varphi(1) = 0; \quad \mu = \mu^s(e), \quad \varphi = \varphi^s(\theta,e)$$
$$\varphi'(0) = \varphi'(1); \quad \mu = \mu^c(e), \quad \varphi = \varphi^c(\theta,e) \tag{2.7}$$

For the approximate analytical solution of problems (2.7), the small-parameter methods described above could be used. However, a detailed analysis turns out to be somewhat longer and more inconvenient, and the coefficients p and r are degenerate. An asymptotic investigation of the solution as $e \to 1$ is particularly difficult, since the functions $\varphi^{s,c}(\theta, e)$ turn out to be unbounded for $\theta=0, 1$. Note that, according to the classical theory of self-adjoint boundary value problems,[16] when $0 \leq e < 1$ there are systems of eigenvalues $\{\mu_n^s\}$, $\{\mu_n^c\}$ and orthonormalized functions $\{\varphi_n^s\}$, $\{\varphi_n^c\}$ with weight r, which form bases.

Hence, we have analytically developed and described the qualitative properties of the resonance zones, unknown in the literature, i.e., regions of exponential instability of the parametric oscillations of a pendulum.

3. Numerical-analytical investigation of the stability of the equilibrium position of a pendulum

The accelerated convergence method[17,18] is extremely effective for analysing the low and moderately high oscillation modes. It contains an algorithm for the highly accurate integration of Cauchy problems for Eq. (1.5) and a procedure for continuation with respect to the parameter e.

Suppose that, for a certain value of $0 \leq e_0 < 1$, we know an approximate solution $\mu_{(0)}$, $u_{(0)}(\theta)$ (for example $\mu_{(0)n}^{s,c}$, $u_{(0)n}^{s,c}(\theta)$ (2.2) when $e_0=0$; the subscript n will henceforth be omitted for brevity). We will take as the criterion of closeness the value of the discrepancy ε_0 from θ for the variable $u_{(0)}^s$ of $u_{(0)}^{c'}$; we will assume that $|\varepsilon_0| \ll 1$, which can be established by a numerical experiment – integration of the following Cauchy problems for known values of $\mu = \mu_{(0)}^{s,c}$ and initial data $u(0)$ and $u'(0)$

$$u'' + (\mu r(\theta,e_0) - e_0 q(\theta,e_0))u = 0$$
$$\mu = \mu_{(0)}^s, \quad u(0) = 0, \quad u'(0) = A^s (= 1)$$
$$\mu = \mu_{(0)}^c, \quad u(0) = A^c (= 1), \quad u'(0) = 0$$
$$r(\theta,e) \equiv (1 - e\cos 2\pi\theta)^{-1}, \quad q(\theta,e) \equiv 4\pi^2 \cos 2\pi\theta\, r(\theta,e) \tag{3.1}$$

As a result of highly accurate integration of Cauchy problems (3.1) the n-th nodes of the functions $u^{s}_{(0)}(\theta)$ and $u^{c'}_{(0)}(\theta)$ are determined. The values of the roots $\xi^{s,c}$ when $\mu^{s,c}_{(0)}$ is fairly close to the accurate values $\mu^{s,c}(e_0)$, turn out to be as close as desired to $\xi = 1 : 1 - \xi_0^{s,c} = \varepsilon_0^{s,c}$, $|\varepsilon_0^{s,c}| \ll 1$. After calculating the discrepancies $\varepsilon_0^{s,c}$ refinement of the eigenvalues is carried out using the final formulae

$$\mu^{s}_{(1)} = \mu^{s}_{(0)} - \varepsilon^{s}_{0}(u'^{s}_{(0)}(1, \mu^{s}_{(0)}))^2 \left\|u^{s}_{(0)}\right\|_r^{-2}$$

$$\mu^{c}_{(1)} = \mu^{c}_{(0)} - \varepsilon^{c}_{0}(\mu^{c}_{(0)} r(1, e_0) - e_0 q(1, e_0))(u^{c}_{(0)})^2(1, \mu^{c}_{(0)}) \left\|u^{c}_{(0)}\right\|_r^{-2}$$

$$\left\|u_{(0)}\right\|_r^2 = \int_0^{\xi_0} u^2_{(0)}(\theta, e_0, \mu_{(0)}) r(\theta, e_0) d\theta \tag{3.2}$$

The values of $\mu^{s,c}_{(1)}$ differ from the exact values $\mu^{s,c}$ by $O(\varepsilon_0^2)$; they are used in the next step of the iterative procedure. Using $\xi_{(1)}$ – the nodes of the functions $u_{(1)}(\theta)$, the discrepancies $\varepsilon_{(1)} = O(\varepsilon^2_{(0)})$ are determined and, by formulae analogous to (3.2), the refined values of $\mu_{(2)}$ are found, which differ from the exact values by $O(\varepsilon^2_{(1)}) = O(\varepsilon_0^4)$. The use of a recurrence procedure based on formulae (3.1) and (3.2) leads to convergence quadrature in ε_0, and enables highly accurate mass calculations to be carried out constructively.[17–19]

The required relations $\mu_n^{s,c}(e)$ are obtained using the procedure of continuation with respect to the parameter: $e = e_0 + \delta e$, where the quantity $\delta e > 0$ is fairly small. The basic and important advantage of the proposed approach compared with variational and functional methods is the individual highly accurate calculation of specific values of $\mu_n(e)$ and $u_n(\theta, e)$. If necessary, one can use different modifications of formulae (3.2) to simplify the calculations.[17,18] For example, for regular behaviour of the functions $u^{s}_{(0)}$ and u'^{c}_{0} close to $\theta = 1$, we can put, with an error $O(\varepsilon_0^2)$,

$$\varepsilon^{s}_{0} u'^{s}_{(0)}(1, \mu^{s}_{(0)}) = u^{s}_{(0)}(1, \mu^{s}_{(0)}), \quad \varepsilon^{c}_{0} u''^{c}_{(0)}(1, \mu^{c}_{(0)}) = u'^{c}_{(0)}(1, \mu^{c}_{(0)}) \tag{3.3}$$

The use of expressions (3.3) for the initial approximation ($i = 0$) and the subsequent approximations ($i \geq 1$) enables us to get rid of the procedure for the highly accurate determination of the discrepancies $\varepsilon_i = 1 - \xi_i$, but requires calculations of $u^{s}_{(i)}$, $u'^{c}_{(i)}$ when $\theta = 1$. The calculation of the norms by quadratures (3.2) can be replaced by integration of the Cauchy problems.[18]

The diagrams shown in the figure describe the limits of the regions of parametric resonance, constructed using the accelerated-convergence method, described extremely briefly above. Inside these regions there is exponential instability of the equilibrium position of a pendulum of variable length. Traditionally, the eigenvalues $\mu_n^{s,c}$ are plotted along the abscissa axis, while the values of the modulation parameter e are plotted along the ordinate axis. The calculations were carried out with a relative error up to the seventh significant digit, i.e., $O(10^{-6})$. To avoid divergence of the curves of the family of eigenvalues $\mu_n^{s,c}$, the quantities $\nu_n^{s,c} = (\mu_n^{s,c}/\pi^2)^{1/2}$ are presented in the figure.

A preliminary analytical investigation of the parametric oscillations were carried out in Section 2 by perturbation theory methods.[2–7] The numerical-graphical results presented above give a much a fuller representation. They confirm that, even for comparatively small values of the modulation index e, there are considerable differences from the traditional curves for the Mathieu equation (the Ince–Strutt and Meissner Diagrams[14,15]). The first resonance zone (see the figure) visually has a finite width. Note that $\mu_1^s \to 0$ and $\mu_1^c \to (2\pi)^2$ when $e \to 1$. The resonance region for the second mode ($n = 2$) is degenerate on the vertical section of the straight line $\mu_2^{s,c}(e) \equiv 4\pi^2$. The corresponding resonance frequency Ω of the change in the length of the pendulum, according to the expression for μ (1.5), is equal to $\Omega_2 = (g/\ell_0)^{1/2}/2$ irrespective of the value of the amplitude ($0 < e < 1$).

The value $\mu = 4\pi^2$ is "attractive" for u_1^c and $\mu_n^{s,c}(n \geq 3)$ as $e \to 1$; these resonance curves (eigenvalues) do not intersect the section of the straight line μ_2 (see the figure). This unique property is not mentioned explicitly in the literature. Obviously it also occurs in the model example for the modified Mathieu equation and in the fundamental Kochin problem on the parametric oscillations of elastic crankshafts.[2,3] The effect requires additional analytical investigations of an asymptotic character (as $e \to 1$).

As mentioned above, the third resonance zone has a small, graphically indistinguishable "width", although the values of μ_3^s and μ_3^c in the region of $e = 1$ may differ considerably (by $O(1)$). For example, when $e = 0.99$ the value of $\mu_3^s \approx 545$ differs from $\mu_3^c \approx 493$ by 5.2. For odd modes ($n = 5, 7, \ldots$) a further narrowing of the resonance zones occurs (similar to the Kochin problem). All the even oscillation modes are degenerate, i.e., the curves of $\mu_{2k}^s(e)$ and $\mu_{2k}^c(e)$ coincide, which is established by locally analytical methods and is confirmed by globally numerical calculations with an error $O(10^{-7})$.

The eigenfunctions $u_n^{s,c}(\theta, e)$ for specified values of $u_n^{s,c}(e)$ and the initial data are obtained by numerical integration of Cauchy problems, similar to (3.1). As $e \to 1$ special methods are required to ensure that the integration process converges with the required accuracy. The accelerated-convergence method enables us to carry out mass calculations and to establish the qualitative analytical properties of the global characteristics of the parametric oscillations of mechanical systems.

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